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Bubbling defect CFT's

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Abstract: We study the gravitational description of conformal half-BPS domain wall operators in $\mathcal{N} = 4$ SYM, which are described by defect CFT's. These defect CFT's arise in the low energy limit of a Hanany-Witten like brane setup and are described in a probe brane approximation by a Karch-Randall brane configuration. The gravitational backreaction takes the five-branes in $AdS_5 \times S^5$ through a geometric transition and turns them into appropriate fluxes which are supported on non-trivial three-spheres.

Keywords: Gauge-gravity correspondence, AdS-CFT Correspondence, Brane Dynamics in Gauge Theories, Intersecting branes models.

Contents

1. Introduction

Local gauge invariant operators are labeled by a point in spacetime. Such operators can be constructed by combining in a gauge invariant way the fields appearing in the action. This is, however, not the only way to define local operators. Operators which cannot be written in a local way in terms of the fields appearing in the action are quite ubiquitous in quantum field theory. Examples of this class of operators include twist operators in conformal field theory and "soliton" operators in gauge theories (see e.g. [1] for a recent discussion). The insertion of an operator in the path integral has the effect of introducing at the location of the operator a very specific singularity for the fields that appear in the action.

There is no essential limitation in quantum field theory restricting the class of admissible singularities of the fundamental fields to be point-like; in principle they can be defined on any defect in spacetime. In four dimensional field theories one may consider line, surface and domain wall operators on top of the more familiar local operators labeled by a point in spacetime. In favorable circumstances — e.g. Wilson loops — these operators can be written down using the fields appearing in the action while in others — e.g. 't Hooft loops — the operators are defined by the singularity they produce for the fundamental fields in the action at the location of the defect.

A convenient way to construct a defect operator is to introduce additional degrees of freedom localized on the defect. The extra degrees of freedom encode the type of singularity produced by the defect operator. The study of defect operators can then by mapped to the problem of studying the defect field theory describing the coupling of a four dimensional field theory to the degrees of freedom living on the defect. Demanding invariance of the defect operator under some symmetry constraints the geometry of the defect as well as the allowed degrees of freedom that can be added to the defect. In particular, demanding invariance under the conformal group on the defect leads to defect conformal field theories [2, 3].

The AdS/CFT conjecture [4–6] requires that all gauge invariant operators in $\mathcal{N}=4$ SYM have a realization in the bulk description. This program has been successfully carried out for the half-BPS local operators in $\mathcal{N}=4$ SYM [6, 7], where the operators can be identified with D-branes in the bulk [8, 9]. Recently, the dictionary has been enlarged [10] (see also [11, 12]) to include all the half-BPS Wilson loop operators in $\mathcal{N}=4$ SYM, which have also been identified with D-branes in the bulk. The half-BPS Wilson loop operators are constructed [10] by integrating out in the defect conformal field theory the localized degrees of freedom living on the loop that are introduced by the bulk D-branes.

In this paper we study half-BPS domain wall operators in $\mathcal{N}=4$ SYM and their corresponding bulk description. Domain wall operators can be identified with a defect conformal field theory describing the coupling of $\mathcal{N}=4$ SYM to additional degrees of freedom localized in an $\mathbb{R}^{1,2}\subset\mathbb{R}^{1,3}$ defect. Supersymmetry requires that the degrees of freedom living on the defect fill three dimensional hypermultiplets. These new localized hypermultiplet degrees of freedom arise from the presence of branes (D5 and NS5-branes) in $AdS_5\times S^5$ that end on a common $\mathbb{R}^{1,2}$ defect at the boundary. For each half-BPS configuration of five-branes in $AdS_5\times S^5$, we may associate a defect conformal field theory or equivalently a half-BPS domain wall operator.

The defect conformal field theory can be derived by studying the low energy effective field theory[1] on N D3-branes in the presence of D5 and NS5-branes intersecting the D3-branes along an $\mathbb{R}^{1,2}\subset\mathbb{R}^{1,3}$ defect. The data which determines the defect conformal field

[1]This brane configuration is a generalization of the brane construction in [13] studied in the context of three dimensional mirror symmetry.

theory under study is the number of D3-branes which end on each of the D5-branes and NS5-branes. In the decoupling/near horizon limit the five-branes span a family of $AdS_4 \times \tilde{S}^2$ and $AdS_4 \times S^2$ geometries in $AdS_5 \times S^5$ respectively, of the type found in [14]. Each such array of five-branes in the bulk corresponds to a half-BPS domain wall operator.

We study the backreaction on the $AdS_5 \times S^5$ background due to the configuration of five-branes dual to a specific domain wall operator. We show that the solution of the supergravity BPS equations is determined by specifying boundary conditions on a two dimensional surface in the ten dimensional geometry. These boundary conditions encode the location where either an S^2 or an $\tilde{S}^2$ shrinks to zero size in a smooth manner. This result generalizes the work of LLM [15] — which applies to the half-BPS local operators — to the geometries dual to half-BPS domain wall[2] operators.

The $AdS_5 \times S^5$ vacuum solution corresponds to an infinite strip, where at the bottom of the strip $\tilde{S}^2$ shrinks to zero size while at the top of the strip S^2 shrinks to zero size. In the probe approximation, a D5-brane is located on the top boundary of the strip while a NS5-brane is located at the bottom. The precise location of brane is determined by the amount of D3-brane charge carried by the five-branes.

In the backreacted geometry we expect these five-branes to be replaced by bubbles of flux [17 − 19], i.e. by smooth non-contractible three-cycles supporting the appropriate amount of three-form flux. There are two ways for this to happen. In one the geometry stays finite and there appears a change in the coloring of the boundary. For a D5-brane this corresponds to having a finite size S^2 and $\tilde{S}^2$ shrinking in a finite segment of the upper boundary. The second possibility is an infinite throat developing on the upper boundary with the S^2 shrunk on both boundaries of the throat. In both cases there appears a smooth three-sphere which can support the three-form flux.

It would be very interesting to get a better understanding of the behavior of the solutions near the defects, how the boundary conditions work and how the fluxes, brane charges and changes in the rank of the gauge group are related to the boundary conditions as we understand them.

The plan of the rest of the paper is as follows. In section 2 we study the brane configuration whose low energy effective field theory yields the defect conformal field theories we are interested in. In section 3 we consider the bulk description of the half-BPS domain wall operators in the probe approximation. In section 4 we derive the BPS equations and some important normalization conditions for spinor bilinears by realizing the (super) symmetry algebra in type IIB supergravity. In section 5 we first discuss the $AdS_5 \times S^5$ solution of the BPS equations, then we 'bootstrap' the general BPS equations to get a second order PDE for one remaining spinor variable and finally we discuss the supersymmetries of probe branes, boundary conditions and the general structure of solutions.

While this paper was in preparation, a paper [20] appeared which overlaps with ours.

2. Half-BPS domain wall operators and defect field theory

Defect operators can be defined by introducing degrees of freedom localized on the defect.

[2] Recently, Yamaguchi [16] has made an analogous ansatz relevant for Wilson loops.

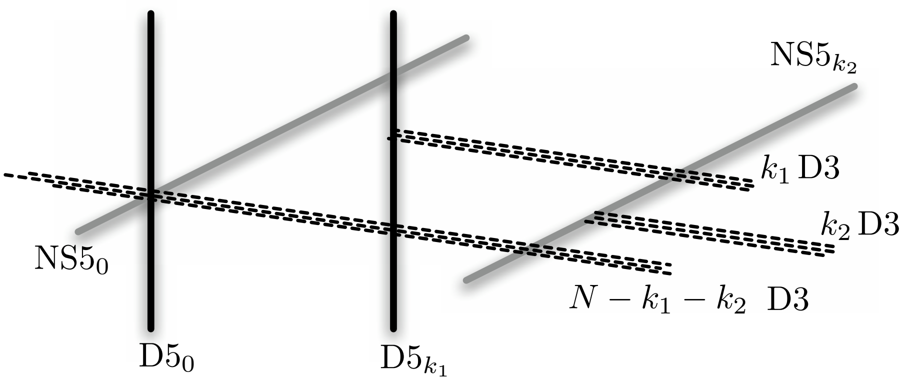

Figure 1: The brane configuration in flat space.

The theory that captures the interactions of the localized degrees of freedom with those of $\mathcal{N} = 4$ SYM is a defect conformal field theory if we impose that the defect preserves conformal invariance.

The description of defect operators in terms of defect conformal field theories naturally suggests the construction of such theories as low energy limits of branes in string theory. The strategy is to consider brane configurations involving D3-branes together with other branes intersecting the D3-branes along a defect. The low energy effective field theory is described by $\mathcal{N} = 4$ SYM coupled to the degrees of freedom localized on the defect introduced by the other branes.

Here we are interested in half-BPS domain wall operators in $\mathcal{N} = 4$ SYM. These operators arise by considering the following brane configuration:

	0	1	2	3	4	5	6	7	8	9
D3	x	x	x	x						
D5	x	x	x		x	x	x			
NS	x	x	x					x	x	x

(2.1)

The degrees of freedom associated with this brane configuration is the $\mathcal{N} = 4$ SYM multiplet together hypermultiplets [13] localized on $\mathbb{R}^{1,2} \subset \mathbb{R}^{1,3}$.

The action for this defect field theory in the absence of NS5-branes and when none of the D3-branes end on the D5-branes has been constructed in [21 − 23]. We refer the reader to these references for the detailed form of the action.

In order to obtain more general domain wall operators, one can allow for configurations where a number of D3-branes end on a D5-brane or NS5-brane. We label by $\mathrm{D5}_k/\mathrm{NS5}_k$ a D5/NS5-brane where k D3-branes end. Each choice of partitioning the N D3-branes among the five-branes corresponds to a different half-BPS domain wall operator. This is similar to the construction of half-BPS Wilson loops [10] using D-branes.

The D3-branes ending on the five-branes have the effect of introducing magnetic charge on the five-brane worlvolume. It would be very interesting to construct explicitly this general class of defect field theories and study in detail the singularities produced on the $\mathcal{N} = 4$ SYM fields by the corresponding half-BPS domain wall operator.

We conclude this section with the analysis of the symmetries of these defect conformal field theories, which play a crucial role in the following sections, where the supergravity description of these operators is studied.

The bosonic symmetry algebra is $\mathrm{SO}(2,3)\times\mathrm{SO}(3)\times\mathrm{SO}(3) = \mathrm{Sp}(4,\mathbb{R})\otimes\mathrm{SO}(4)$, which is generated by generators M^A and M^L. In the $(4,4)$ those generators $(M^A)^{\alpha}{}_{\beta}$ and $(M^L)^{\dot\alpha}{}_{\dot\beta}$ can be chosen real and the supersymmetry generators[3] $Q_{\alpha\dot\alpha}$ are Hermitean

$$Q^{\dagger}_{\alpha\dot\alpha} = Q_{\alpha\dot\alpha}, \tag{2.2}$$

they transform as

$$[M^A, Q_{\alpha\dot\alpha}] = -(M^A)^{\beta}{}_{\alpha} Q_{\beta\dot\alpha} \qquad \text{and} \qquad [M^L, Q_{\alpha\dot\alpha}] = -(M^L)^{\dot\beta}{}_{\dot\alpha} Q_{\alpha\dot\beta}. \tag{2.3}$$

Their anti commutation relation is

$$\{Q_{\alpha\dot\alpha}, Q_{\beta\dot\beta}\} = i(M_A J)_{\alpha\beta} I_{\dot\alpha\dot\beta} M^A - i J_{\alpha\beta}(M_L I)_{\dot\alpha\dot\beta} M^L, \tag{2.4}$$

where $J_{\alpha\beta}$ is the real, invariant, antisymmetric matrix of $\mathrm{Sp}(4,\mathbb{R})$ and $I_{\dot\alpha\dot\beta}$ is the real, invariant, symmetric matrix of $\mathrm{SO}(4)$. Therefore, these defect conformal field theories are invariant under an $\mathrm{OSp}(4|4)$ subalgebra of the $\mathrm{SU}(2,2|4)$ algebra of $\mathcal{N}=4$ SYM.

The supersymmetry generators can be contracted with real Grassmann variables $\hat\epsilon_{\alpha\dot\alpha}$ to form Hermitean generators

$$\hat\epsilon^{\alpha\dot\alpha} Q_{\alpha\dot\alpha} = \hat\epsilon^{\alpha\dot\alpha *} Q^{\dagger}_{\alpha\dot\alpha}. \tag{2.5}$$

3. Probe branes in $AdS_5 \times S^5$

In this section we study in the probe approximation the branes in $AdS_5 \times S^5$ which correspond to the half-BPS domain wall operators described in the previous section. In the next sections we study the backreaction produced by these branes and find the equations which determine the asymptotically AdS geometries dual to the defect operators.

In order to make manifest the $\mathrm{SO}(2,3)\otimes\mathrm{SO}(3)\otimes\mathrm{SO}(3)$ symmetry of the half-BPS domain wall operators we foliate the AdS_5 geometry by AdS_4 slicings

$$ds^2 = R^2\left(\cosh^2(x) ds_{AdS_4} + dx^2\right), \tag{3.1}$$

where R is the radius of curvature of AdS_5 and S^5. We also foliate S^5 by $S^2 \times \tilde S^2$ slicings

$$ds^2 = R^2\left(dy^2 + \cos^2(y) d\Omega_2 + \sin^2(y) d\tilde\Omega_2\right), \tag{3.2}$$

where $d\Omega_2(d\tilde\Omega_2)$ is the metric on a unit S^2 ($\tilde S^2$).

We note that in this parametrization the $AdS_5 \times S^5$ metric can be represented by an $AdS_4 \times S^2 \times \tilde S^2$ fibration over a strip, whose length is parametrized by x and width by y. At the $y=0$ boundary of the strip $\tilde S^2$ shrinks smoothly to zero size while at the $y=\pi/2$ boundary S^2 shrinks smoothly.

[3] The undotted index $\alpha = 1,\ldots,4$ is an index in the real fundamental representation of $\mathrm{Sp}(4,\mathbb{R})$, whereas the dotted index $\dot\alpha = 1,\ldots,4$ is an index in the real fundamental representation of $\mathrm{SO}(4)$.

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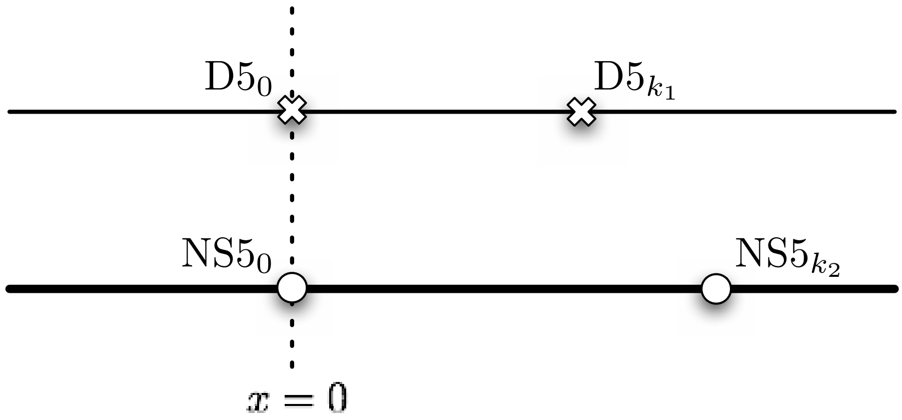

Figure 2: Probe branes in $AdS_5 \times S^5$.

The $D5_k$-brane in the previous section becomes in the near horizon limit a D5-brane in $AdS_5 \times S^5$ with an $AdS_4 \times \tilde{S}^2$ worldvolume and with k units of magnetic flux dissolved on the D5-brane. The details of this solution can be found in [21] and the analysis of supersymmetry in [24]. A $D5_k$-brane sits at $y = \pi/2$ and at $x(k) = \sinh^{-1}(\pi k/R)$.

Similarly, the solution for the $NS5_k$-brane can also be found. It spans an $AdS_4 \times S^2$ geometry in $AdS_5 \times S^5$ and has k units of magnetic flux dissolved in it. Now, the $NS5_k$-brane sits at $y = 0$ and at $x(k) = \sinh^{-1}(\pi k/R)$.

We will show in section 5.3 that all these five-branes preserve exactly the same supersymmetries and coincide with the supersymmetries preserved by the defect conformal field theory.

We therefore see that any half-BPS domain wall operator in the probe approximation can be characterized by a collection of points on the appropriate boundary of the strip which characterizes $AdS_5 \times S^5$. To each $D5_k$-brane of the microscopic description of the defect conformal field theory we associate a point at the $y = \pi/2$ boundary of the strip located at $x(k)$, where k is the number of D3-branes ending on the D5-brane. Similarly, to each $NS5_k$-brane of the microscopic description of the defect conformal field theory we associate a point at the $y = 0$ boundary of the strip located at $x(k)$, where k is the number of D3-branes ending on the NS5-brane. Therefore, to a given half-BPS domain wall operator we can associate the following strip

The goal of the rest of the paper is to find the BPS equations in Type IIB supergravity which determine the backreaction[4] produced by a collection of five-branes corresponding dual to a defect conformal field theory.

4. The supergravity solution

In this section we derive the BPS equations by reproducing the OSp(4|4) supersymmetry algebra of the half-BPS domain wall operators in Type IIB supergravity. This consists of two parts, the invariance of the background under the (super) symmetry transformations

[4] In [25] the supergravity equations for intersecting D3/D5 branes were analyzed.

as well as the closure of the (super) symmetry algebra. This section is very technical. Readers who do not want to go into technical details can just read 4.1. The result of this section is the BPS equations (4.47), (4.48) and (4.50) together with the normalization conditions (4.38). We are using the type IIB supergravity conventions of [26, 27] with a mostly + signature. The gamma matrix conventions are summarized in appendix A.

4.1 The ansatz

The bosonic symmetry group is SO(2, 3) × SO(3) × SO(3) and the 10 dimensional space time is a $AdS_4 \times S^2 \times S^2$ fibration over a two dimensional base space M_2. The most general vielbein ansatz is

$$\begin{aligned} &e^\mu = A_1\, e^{\hat{\mu}}, &&\mu = 0,1,2,3,\\ &e^m = A_2\, e^{\tilde{m}}, &&m = 4,5,\\ &e^i = A_3\, e^{\check{i}}, &&i = 6,7,\\ &e^a, &&a = 8,9, \end{aligned} \tag{4.1}$$

where $e^{\hat{\mu}}$ is a vielbein on a unit AdS_4, $e^{\tilde{m}}$ and $e^{\check{i}}$ are vielbeins on the unit S^2 and $\tilde{S}^2$ respectively and e^a is a vielbein on M_2. The most general self dual 5-form flux has the form

$$F = f_a(e^{0123a} + \epsilon_{ab}\, e^{4567b}), \tag{4.2}$$

where $f_a\, e^a$ is a real 2-form on M_2. The most general dilaton-axion P and 3-form fluxes G are given in terms of the complex 1-forms $p^{(4)} = p_a\, e^a$, $g^{(4)} = g_a\, e^a$ and $h^{(4)} = h_a\, e^a$ on M_2

$$P = p_a\, e^a \qquad \text{and} \qquad G = g_a\, e^{45a} + ih_a\, e^{67a}. \tag{4.3}$$

The most general U(1)-R connection is given by the two dimensional connection $q^{(4)} = q_a\, e^a$ on M_2

$$Q = q_a\, e^a. \tag{4.4}$$

4.2 A specific basis for OSp(4|4)

We will derive the BPS equations by heavily using the explicit form of the symmetry algebra OSp(4|4) in terms of the clifford algebras of SO(2, 3) × SO(3) × SO(3). We use the Clifford algebra conventions of appendix A. A basis of generators in the 4 of $\mathrm{Sp}(4,\mathbb{R})$ is

$$M^{\mu\nu} \sim \frac{1}{2}\gamma^{\mu\nu}, \qquad M^\mu \sim \frac{1}{2}\gamma^\mu. \tag{4.5}$$

The matrix

$$J = iD^{(1)}\gamma^{(1)} \tag{4.6}$$

is invariant and antisymmetric. One can coose a basis of Majorana spinors

$$\chi^*_\alpha = B^{(1)}\chi_\alpha \tag{4.7}$$

and a dual basis of spinors χ^α such that

$$\chi^{\alpha t}\chi_\beta = \delta^\alpha_\beta. \tag{4.8}$$

Then

$$(M^{\mu\nu})^{\alpha}{}_{\beta} = \chi^{\alpha t} M^{\mu\nu} \chi_{\beta} \qquad \text{and} \qquad (M^{\mu})^{\alpha}{}_{\beta} = \chi^{\alpha t} M^{\mu} \chi_{\beta} \tag{4.9}$$

are real and

$$J_{\alpha\beta} = \chi^{t}_{\alpha} J \chi_{\beta} \tag{4.10}$$

is real, invariant and antisymmetric.

Similarly,

$$\begin{array}{l} M^{mn} \sim \frac{1}{2}(\gamma^{mn} \otimes \mathbb{1}),\ M^{m} \sim \frac{i}{2}(\gamma^{m} \otimes \mathbb{1}), \\ M^{ij} \sim \frac{1}{2}(\mathbb{1} \otimes \gamma^{ij}), \quad\ M^{i} \sim \frac{i}{2}(\mathbb{1} \otimes \gamma^{i}) \end{array} \tag{4.11}$$

are a basis of generators of the 4 of SO(4), the matrix

$$I = D^{(2)} \otimes D^{(3)} \tag{4.12}$$

is invariant and symmetric. One can coose a basis of Majorana spinors

$$\chi^{*}_{\dot\alpha} = (B^{(2)} \otimes B^{(3)}) \chi_{\dot\alpha} \tag{4.13}$$

and a dual basis of spinors $\chi^{\dot\alpha}$ such that

$$\chi^{\dot\alpha t} \chi_{\dot\beta} = \delta^{\dot\alpha}_{\dot\beta}. \tag{4.14}$$

Then $(M^{mn})^{\dot\alpha}{}_{\dot\beta}$,$(M^{m})^{\dot\alpha}{}_{\dot\beta}$, $(M^{ij})^{\dot\alpha}{}_{\dot\beta}$ and $(M^{i})^{\dot\alpha}{}_{\dot\beta}$ are real and $I_{\dot\alpha\dot\beta}$ is real, invariant and symmetric.

The anticommutation relation of the supercharges is then

$$\begin{aligned} \{Q_{\alpha\dot\alpha}, Q_{\beta\dot\beta}\} = {} & \tfrac{1}{4}(\bar\chi_{\alpha\dot\alpha}(\gamma^{(1)}\gamma_{\mu\nu} \otimes \mathbb{1} \otimes \mathbb{1})\chi_{\beta\dot\beta})M^{\mu\nu} + \tfrac{1}{2}(\bar\chi_{\alpha\dot\alpha}(\gamma^{(1)}\gamma_{\mu} \otimes \mathbb{1} \otimes \mathbb{1})\chi_{\beta\dot\beta})M^{\mu} - \\ & \tfrac{1}{4}(\bar\chi_{\alpha\dot\alpha}(\gamma^{(1)} \otimes \gamma_{mn} \otimes \mathbb{1})\chi_{\beta\dot\beta})M^{mn} - \tfrac{i}{2}(\bar\chi_{\alpha\dot\alpha}(\gamma^{(1)} \otimes \gamma_{m} \otimes \mathbb{1})\chi_{\beta\dot\beta})M^{m} - \\ & \tfrac{1}{4}(\bar\chi_{\alpha\dot\alpha}(\gamma^{(1)} \otimes \mathbb{1} \otimes \gamma_{ij})\chi_{\beta\dot\beta})M^{ij} - \tfrac{i}{2}(\bar\chi_{\alpha\dot\alpha}(\gamma^{(1)} \otimes \mathbb{1} \otimes \gamma_{i})\chi_{\beta\dot\beta})M^{i}. \end{aligned} \tag{4.15}$$

4.3 Symmetries and the Killing spinor equations

The Killing spinors ϵ have to transform in the $(4,2,2)$ representation of the Bosonic symmetry group $\mathrm{SO}(2,3) \times \mathrm{SO}(3) \times \mathrm{SO}(3)$. The Bosonic symmetries are realized by Killing vector fields. Those act through the Lie derivative on the Killing spinors.

For a given point Q on M_{10} there is a $\mathrm{SO}(1,3) \times \mathrm{SO}(2) \times \mathrm{SO}(2)$ stabilizer group. The Lie derivative for this stabilizer group acts by rotations

$$\frac{1}{2}\gamma^{\mu\nu}, \qquad \frac{1}{2}\gamma^{mn} \qquad \text{and} \qquad \frac{1}{2}\gamma^{ij} \tag{4.16}$$

on the Killing spinor ϵ at Q. For the tangent vectors $e_{\hat\mu}$, $e_{\tilde m}$ and $e_{\check i}$ at Q there are unique Killing vector fields which generate a geodesic through Q in the fiber. The Lie derivative along those Killing vector fields at Q are given by the covariant derivatives

$$\mathcal{L}_{\hat\mu}\epsilon = \nabla_{\hat\mu}\epsilon, \qquad \mathcal{L}_{\tilde m}\epsilon = \nabla_{\tilde m}\epsilon \qquad \text{and} \qquad \mathcal{L}_{\check i}\epsilon = \nabla_{\check i}\epsilon. \tag{4.17}$$

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The fact that the Killing spinors ϵ have to transform in the $(4,2,2)$ representation of the Bosonic symmetry group $\mathrm{SO}(2,3)\times\mathrm{SO}(3)\times\mathrm{SO}(3)$ implies that the Lie derivative action on a Killing spinor can be reproduced by a matrix action N_μ, N_m and N_i which is consistent with (4.16), (4.5) and (4.9). Consistency and the 10-dimensional chirality condition then imply that

$$\begin{aligned} N_\mu &= \tfrac{1}{2}n(\gamma_\mu\otimes\mathbb{1}\otimes\mathbb{1}\otimes\gamma_8)n^{-1},\\ N_m &= \tfrac{i}{2}n(\mathbb{1}\otimes\gamma_m\otimes\mathbb{1}\otimes\gamma_8)n^{-1} \qquad \text{and}\\ N_i &= \tfrac{i}{2}n(\mathbb{1}\otimes\mathbb{1}\otimes\gamma_i\otimes\gamma_8)n^{-1},\end{aligned} \tag{4.18}$$

where n is a unitary matrix of the form

$$n = f_{\eta_1\eta_2\eta_3\eta_4}(\gamma^{(1)})^{\frac{2-\eta_1}{2}}(\gamma^{(2)})^{\frac{2-\eta_2}{2}}(\gamma^{(3)})^{\frac{2-\eta_3}{2}}(\gamma^{(4)})^{\frac{2-\eta_4}{2}}. \tag{4.19}$$

Note that (4.18) does not fix n uniquely. The Killing spinor equations then have the form

$$\begin{aligned}(\nabla_{\hat\mu}-N_\mu)\epsilon&=0,\\ (\nabla_{\tilde m}-N_m)\epsilon&=0,\\ (\nabla_{\check i}-N_i)\epsilon&=0.\end{aligned} \tag{4.20}$$

To solve those 10-dimensional Killing spinor equations let us first have a look at the simplified 8-dimensional and 2-dimensional Killing spinor equations

$$\begin{aligned}(\nabla_{\hat\mu}-\tfrac{\eta_1}{2}(\gamma_\mu\otimes\mathbb{1}\otimes\mathbb{1}))\chi^{(\eta_1,\eta_2,\eta_3)}_{\alpha\dot\alpha}&=0,\\ (\nabla_{\tilde m}-\tfrac{i\eta_2}{2}(\mathbb{1}\otimes\gamma_m\otimes\mathbb{1}))\chi^{(\eta_1,\eta_2,\eta_3)}_{\alpha\dot\alpha}&=0,\\ (\nabla_{\check i}-\tfrac{i\eta_3}{2}(\mathbb{1}\otimes\mathbb{1}\otimes\gamma_i))\chi^{(\eta_1,\eta_2,\eta_3)}_{\alpha\dot\alpha}&=0,\\ (\mathbb{1}-\eta_4\gamma^8)\zeta^{(\eta_4)}&=0.\end{aligned} \tag{4.21}$$

The solutions to those equations transform in the $(4,2,2)$ of $\mathrm{SO}(1,3)\times\mathrm{SO}(2)\times\mathrm{SO}(2)$. This representation allows for a reality condition. A basis of the real representation is labelled by $(\alpha,\dot\alpha)$

$$(B^{(1)}\otimes B^{(2)}\otimes B^{(3)})^{-1}(\chi^{(1,1,1)}_{\alpha\dot\alpha})^* = \chi^{(1,1,1)}_{\alpha\dot\alpha}. \tag{4.22}$$

Define

$$\epsilon_0 = \hat\epsilon^{\alpha\dot\alpha}\chi^{(1,1,1)}_{\alpha\dot\alpha}\otimes\zeta^{(1)}, \tag{4.23}$$

where $\hat\epsilon^{\alpha\dot\alpha}$ is a real Grassman variable defined in the last section. It is not hard to see that

$$\epsilon = n\epsilon_0 = \hat\epsilon^{\alpha\dot\alpha}f_{\eta_1\eta_2\eta_3\eta_4}\chi^{(\eta_1,\eta_2,\eta_3)}_{\alpha\dot\alpha}\otimes\zeta^{(\eta_4)} = \hat\epsilon^{\alpha\dot\alpha}\chi^{(\eta_1,\eta_2,\eta_3)}_{\alpha\dot\alpha}\otimes\zeta_{\eta_1\eta_2\eta_3} \tag{4.24}$$

is the general solution of (4.20).

The 10-dimensional chirality condition imposes

$$\gamma^{(4)}\zeta_{-\eta} = \zeta_\eta, \tag{4.25}$$

whereas the reality condition implies

$$*\epsilon = -\eta_1\eta_2\eta_3\hat\epsilon^{\alpha\dot\alpha}\chi^{(\eta_1,\eta_2,\eta_3)}_{\alpha\dot\alpha}\otimes(*\zeta_{-\eta_1,-\eta_2,\eta_3}). \tag{4.26}$$

This allows to translate the dilatino and gravitino variation equations into equations which depend linearly on

$$\hat\epsilon^{\alpha\dot\alpha}\chi^{(\eta_1,\eta_2,\eta_3)}_{\alpha\dot\alpha}. \tag{4.27}$$

4.4 The dilatino and gravitino variation equations

The dilatino variation equation

$$P_M\,\gamma^M * \epsilon + \frac{1}{24}\,G_{MNP}\gamma^{MNP}\epsilon = 0 \tag{4.28}$$

turns into[5]

$$ip_a\sigma^{(1,1,0)}\gamma^a * \zeta - \frac{ig_a}{24}\sigma^{(1,0,1)}\gamma^a\zeta + \frac{h_a}{24}\sigma^{(1,1,0)}\gamma^a\zeta = 0. \tag{4.29}$$

To calculate the gravitino variation equations

$$\mathcal{D}_M\epsilon + \frac{i}{480}F_{PQRST}\gamma^{PQRST}\gamma_M\epsilon - \frac{1}{96}G_{PQR}({\gamma_M}^{PQR} - 9\delta_M^P\gamma^{QR}) * \epsilon = 0, \tag{4.30}$$

we need the spin connection

$$\begin{array}{ll} \omega_{\mu\nu\rho} = \frac{1}{A_1}\omega_{\hat\mu\hat\nu\hat\rho}, & \omega_{\mu\nu a} = \eta_{\mu\nu}\frac{\partial_a A_1}{A_1}, \\ \omega_{mnp} = \frac{1}{A_2}\omega_{\tilde m\tilde n\tilde p}, & \omega_{mna} = \delta_{mn}\frac{\partial_a A_2}{A_2}, \\ \omega_{ijk} = \frac{1}{A_3}\omega_{\check i\check j\check k}, & \omega_{ija} = \delta_{ij}\frac{\partial_a A_3}{A_3}, \\ \omega_{aab}. & \end{array} \tag{4.31}$$

The gravitino variation equations turn into

$$\begin{aligned} \frac{i}{2A_1}\sigma^{(2,1,1)}\zeta + \frac{\partial_a A_1}{2A_1}\gamma^a\zeta + \frac{f_a}{240}\gamma^a\sigma^{(1,0,0)}\zeta - \frac{g_a}{96}\gamma^a\sigma^{(0,1,1)} * \zeta - \frac{ih_a}{96}\gamma^a * \zeta &= 0,\\ -\frac{1}{2A_2}\sigma^{(0,2,1)}\zeta + \frac{\partial_a A_2}{2A_2}\gamma^a\zeta - \frac{f_a}{240}\gamma^a\sigma^{(1,0,0)}\zeta + \frac{g_a}{32}\gamma^a\sigma^{(0,1,1)} * \zeta - \frac{ih_a}{96}\gamma^a * \zeta &= 0,\\ -\frac{1}{2A_3}\sigma^{(0,0,2)}\zeta + \frac{\partial_a A_3}{2A_3}\gamma^a\zeta - \frac{f_a}{240}\gamma^a\sigma^{(1,0,0)}\zeta - \frac{g_a}{96}\gamma^a\sigma^{(0,1,1)} * \zeta + \frac{ih_a}{32}\gamma^a * \zeta &= 0,\\ \mathcal{D}_a\zeta + \frac{f_b}{240}\gamma^b\gamma_a\sigma^{(1,0,0)}\zeta - \frac{g_b}{96}{\gamma_a}^b\sigma^{(0,1,1)} * \zeta + \frac{g_a}{32}\sigma^{(0,1,1)} * \zeta - \frac{ih_b}{96}{\gamma_a}^b * \zeta + \frac{ih_a}{32} * \zeta &= 0 \end{aligned}$$

4.5 Killing vectors, Lorentz rotations and constraints on spinor bilinears

The supersymmetry algebra implies that the spinor bilinears

$$\xi^M = \mathrm{Re}(\bar\epsilon_1\gamma^M\epsilon_2) \tag{4.32}$$

are Killing vectors which generate the Bosonic symmetries that appear in the anticommutator (4.15) of the two supersymmetries generated by ϵ_1 and ϵ_2 [28]. This implies the equations

$$\begin{aligned} -2\,\mathrm{Im}(\bar\epsilon_1\gamma^\mu\epsilon_2) &= \tfrac{iA_1}{2}\hat\epsilon_1^{\alpha\dot\alpha}\hat\epsilon_2^{\beta\dot\beta}\left(\bar\chi^{(1,1,1)}_{\alpha\dot\alpha}((\gamma^{(1)}\gamma^\mu)\otimes\mathbb{1}\otimes\mathbb{1})\chi^{(1,1,1)}_{\beta\dot\beta}\right),\\ -2\,\mathrm{Im}(\bar\epsilon_1\gamma^m\epsilon_2) &= \tfrac{A_2}{2}\hat\epsilon_1^{\alpha\dot\alpha}\hat\epsilon_2^{\beta\dot\beta}\left(\bar\chi^{(1,1,1)}_{\alpha\dot\alpha}(\gamma^{(1)}\otimes\gamma^m\otimes\mathbb{1})\chi^{(1,1,1)}_{\beta\dot\beta}\right),\\ -2\,\mathrm{Im}(\bar\epsilon_1\gamma^i\epsilon_2) &= \tfrac{A_3}{2}\hat\epsilon_1^{\alpha\dot\alpha}\hat\epsilon_2^{\beta\dot\beta}\left(\bar\chi^{(1,1,1)}_{\alpha\dot\alpha}(\gamma^{(1)}\otimes\mathbb{1}\otimes\gamma^i)\chi^{(1,1,1)}_{\beta\dot\beta}\right),\\ -2\,\mathrm{Im}(\bar\epsilon_1\gamma^a\epsilon_2) &= 0. \end{aligned} \tag{4.33}$$

[5] Note that $*\zeta = \gamma^8\sigma^{(2,2,2)}\zeta^*$ is the covariant complex conjugation.

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Furthermore the Lorentz rotation that appears in the anticommutator of the two supersymmetries generated by ϵ_1 and ϵ_2 is

$$\begin{aligned} l^{MN} = \omega_P{}^{MN}\xi^P - \tfrac{1}{3}F^{MNPQR}\,\mathrm{Re}(\bar\epsilon_1\gamma_{PQR}\epsilon_2) - \\ \tfrac{3}{4}\,\mathrm{Im}\left(G^{MNP}\bar\epsilon_1\gamma_P * \epsilon_2 - \tfrac{1}{18}G_{PQR}\bar\epsilon_1\gamma^{MNPQR} * \epsilon_2\right). \end{aligned} \tag{4.34}$$

Comparison with (4.15) leads to

$$\begin{aligned} l^{\mu\nu} &= \tfrac{i}{4}\hat\epsilon_1^{\alpha\dot\alpha}\hat\epsilon_2^{\beta\dot\beta}\left(\bar\chi^{(1,1,1)}_{\alpha\dot\alpha}((\gamma^{(1)}\gamma^{\mu\nu})\otimes\mathbb{1}\otimes\mathbb{1})\chi^{(1,1,1)}_{\beta\dot\beta}\right), \\ l^{mn} &= -\tfrac{i}{4}\hat\epsilon_1^{\alpha\dot\alpha}\hat\epsilon_2^{\beta\dot\beta}\left(\bar\chi^{(1,1,1)}_{\alpha\dot\alpha}((\gamma^{(1)}\otimes\gamma^{mn})\otimes\mathbb{1})\chi^{(1,1,1)}_{\beta\dot\beta}\right), \\ l^{ij} &= -\tfrac{i}{4}\hat\epsilon_1^{\alpha\dot\alpha}\hat\epsilon_2^{\beta\dot\beta}\left(\bar\chi^{(1,1,1)}_{\alpha\dot\alpha}((\gamma^{(1)}\otimes\mathbb{1}\otimes\gamma^{ij}))\chi^{(1,1,1)}_{\beta\dot\beta}\right) \end{aligned} \tag{4.35}$$

with all other Lorentz rotations vanishing.

The left hand sides of (4.33) can be expanded using (4.24), the identities (4.24) and the symmetry properties (B.5). This implies[6]

$$\begin{aligned} &\bar\zeta\zeta = A_1, \qquad \bar\zeta\sigma^{(1,1,0)}\zeta = \bar\zeta\sigma^{(1,0,1)}\zeta = \bar\zeta\sigma^{(0,1,1)}\zeta = 0, \\ &\bar\zeta\sigma^{(2,3,0)}\zeta = A_2,\ \bar\zeta\sigma^{(3,3,0)}\zeta = \bar\zeta\sigma^{(2,2,0)}\zeta = \bar\zeta\sigma^{(3,2,0)}\zeta = 0, \\ &\bar\zeta\sigma^{(2,1,3)}\zeta = A_3,\ \bar\zeta\sigma^{(3,1,3)}\zeta = \bar\zeta\sigma^{(2,1,2)}\zeta = \bar\zeta\sigma^{(3,1,2)}\zeta = 0, \\ &\bar\zeta\gamma^a\sigma^{(3,1,0)}\zeta = \bar\zeta\gamma^a\sigma^{(3,0,1)}\zeta = \bar\zeta\gamma^a\sigma^{(2,1,0)}\zeta = \bar\zeta\gamma^a\sigma^{(2,0,1)}\zeta = 0. \end{aligned} \tag{4.36}$$

Using the chirality condition (4.25) one can see that in the last set of equations one can set $a = 8$. This leaves 16 real equations for the 16 real components of ζ. The overall phase of ζ cannot be determined from those equations. This means that the system of equations is overdetermined by one equation. Doing a cyclic permutation of the Pauli matrices $(0,1,2,3) \to (0,3,1,2)$ the above equations are solved by

$$\begin{aligned} &f_{1,1,1,1} = e^{i\phi}\alpha, \qquad f_{1,1,-1,1} = i\eta_1 e^{i\phi}\alpha^*, \\ &f_{1,-1,1,1} = -\eta_1\eta_2 e^{i\phi}\alpha^*,\ f_{1,-1,-1,1} = i\eta_2 e^{i\phi}\alpha, \\ &f_{-1,1,1,1} = -i\eta_2 e^{i\phi}\beta, \quad f_{-1,1,-1,1} = -\eta_1\eta_2 e^{i\phi}\beta^*, \\ &f_{-1,-1,1,1} = -i\eta_1 e^{i\phi}\beta^*,\ f_{-1,-1,-1,1} = e^{i\phi}\beta \end{aligned} \tag{4.37}$$

with

$$|\alpha|^2 + |\beta|^2 = \frac{A_1}{16}, \qquad \alpha\beta = \frac{\nu_1(A_2 - i\nu_2 A_3)}{32}. \tag{4.38}$$

This leaves ϕ and the relative phase of α and β undetermined. From now on we will continue working in the basis with cyclically permuted Pauli matrices.

The conditions (4.34) for $(MN) = (\mu m)$ and $(MN) = (\mu i)$ are

$$\mathrm{Im}\left(g_a\bar\epsilon_1\gamma^{\mu 45ia} * \epsilon_2\right) = \mathrm{Re}\left(h_a\bar\epsilon_1\gamma^{\mu m67a} * \epsilon_2\right) = 0. \tag{4.39}$$

These imply the reality conditions

$$\mathrm{Re}(e^{-2i\phi}g_a) = \mathrm{Re}(e^{-2i\phi}h_a) = 0 \tag{4.40}$$

or

$$g_a^* = -e^{-4i\phi}g_a \qquad \text{and} \qquad h_a^* = -e^{-4i\phi}h_a. \tag{4.41}$$

The other conditions from the closure of the supersymmetry algebra are more involved and we do not need them.

[6]The σ-s are Pauli matrices acting on the η-indices. Here $\sigma^0 = \mathbb{1}$ and $\sigma^{(i,j,k)} = \sigma^i\otimes\sigma^j\otimes\sigma^k$.

4.6 The BPS equations

We can insert the results of the last section into the Gravitino and Dilatino variation equations. The dilatino variation equations impose a reality condition on P

$$(p_8^* - ip_9^*) = e^{-8i\phi}(p_8 - ip_9) \tag{4.42}$$

one can gauge fix the U(1) R-symmetry of type IIB supergravity by demanding $e^{2i\phi} = i$. Then the reality conditions read

$$g_a^* = g_a, \qquad h_a^* = h_a \qquad \text{and} \qquad (p_8^* - ip_9^*) = (p_8 - ip_9). \tag{4.43}$$

To derive the remaining BPS equations it is useful to fix reparametrization invariance by going to the conformally flat metric on M_2

$$e^8 = A_4\, dx \qquad \text{and} \qquad e^9 = A_4\, dy, \tag{4.44}$$

and introducing the complex coordinate z by

$$dz = dx + idy. \tag{4.45}$$

It is useful to combine the real 1-forms f_a, g_a, h_a and p_a into

$$\begin{aligned} f &= \tfrac{A_4}{2}(f_8 - if_9), & g &= \tfrac{A_4}{2}(f_8 - if_9), \\ h &= \tfrac{A_4}{2}(h_8 - ih_9), & \text{and } p &= \tfrac{A_4}{2}(p_8 - ip_9). \end{aligned} \tag{4.46}$$

The dilatino variation equations then give rise to the BPS equations

$$\begin{aligned} p\beta^* + \tfrac{1}{24}(g + ih)\alpha &= 0, \\ p\alpha - \tfrac{1}{24}(g - ih)\beta^* &= 0. \end{aligned} \tag{4.47}$$

The Gravitino variation equations in the μ, m and i directions give rise to the BPS equations

$$\begin{aligned} &\tfrac{\nu_2 A_4}{2A_1}\beta + \tfrac{\partial_z A_1}{A_1}\alpha + \tfrac{f}{120}\alpha + \tfrac{g+ih}{48}\beta^* = 0, \\ &\tfrac{\nu_2 A_4}{2A_1}\alpha^* - \tfrac{\partial_z A_1}{A_1}\beta^* + \tfrac{f}{120}\beta^* + \tfrac{g-ih}{48}\alpha = 0, \\ &-\tfrac{\nu_1\nu_2 A_4}{2A_2}\alpha^* + \tfrac{\partial_z A_2}{A_2}\alpha - \tfrac{f}{120}\alpha - \tfrac{3g-ih}{48}\beta^* = 0, \\ &-\tfrac{\nu_1\nu_2 A_4}{2A_2}\beta - \tfrac{\partial_z A_2}{A_2}\beta^* - \tfrac{f}{120}\beta^* - \tfrac{3g+ih}{48}\alpha = 0, \\ &\tfrac{i\nu_1 A_4}{2A_3}\alpha^* + \tfrac{\partial_z A_3}{A_3}\alpha - \tfrac{f}{120}\alpha + \tfrac{g-3ih}{48}\beta^* = 0, \\ &-\tfrac{i\nu_1 A_4}{2A_3}\beta - \tfrac{\partial_z A_3}{A_3}\beta^* - \tfrac{f}{120}\beta^* + \tfrac{g+3ih}{48}\alpha = 0. \end{aligned} \tag{4.48}$$

Finally, the gravitino variation equations in the a-direction give rise to the reality condition

$$q_a = \partial_a \phi, \tag{4.49}$$

which reduces to $q_a = 0$ in the chosen gauge, together with the BPS equations

$$\begin{aligned} &\partial_z\alpha + \tfrac{\partial_z A_4}{2A_4}\alpha - \tfrac{g+ih}{48}\beta^* = 0, \\ &\partial_z\beta^* + \tfrac{\partial_z A_4}{2A_4}\beta^* + \tfrac{g-ih}{48}\alpha = 0, \\ &\partial_z\alpha^* - \tfrac{\partial_z A_4}{2A_4}\alpha^* + \tfrac{f}{120}\alpha^* - \tfrac{g-ih}{24}\beta = 0, \\ &\partial_z\beta - \tfrac{\partial_z A_4}{2A_4}\beta - \tfrac{f}{120}\beta + \tfrac{g+ih}{24}\alpha^* = 0. \end{aligned} \tag{4.50}$$

4.7 Bianchi identities

The Bianchi identities

$$\begin{aligned}\mathcal{D}P &= 0,\\ \mathcal{D}G &= -P \wedge G^*,\\ dQ &= -iP \wedge P^*,\\ dF &= \tfrac{5i}{12} G \wedge G^*\end{aligned} \tag{4.51}$$

turn into the equations

$$\begin{aligned}dp^{(4)} &= 0,\\ d(A_2^2\, g^{(4)}) + p^{(4)} \wedge (A_2^2\, g^{(4)}) &= 0,\\ d(A_3^2\, h^{(4)}) - p^{(4)} \wedge (A_3^2\, h^{(4)}) &= 0,\\ p^{(4)} \wedge p^{(4)*} &= 0,\\ d(A_1^4\, f^{(4)}) &= 0,\\ d(A_2^2\, A_3^2\, * f^{(4)}) &= \tfrac{5}{6}\, A_2^2\, A_3^2\, g^{(4)} \wedge h^{(4)}.\end{aligned} \tag{4.52}$$

The first four identities can be solved by introducing the functions ρ, l, m and n

$$\begin{aligned}p^{(4)} &= d\rho,\\ g^{(4)} &= \tfrac{e^{-\rho}}{A_2^2}\, dm,\\ h^{(4)} &= \tfrac{e^{\rho}}{A_3^2}\, dn,\\ f^{(4)} &= \tfrac{1}{A_1^4}\, dl,\end{aligned} \tag{4.53}$$

the last equation leads to a harmonic equation for l.

5. The domain wall geometries

In this section we will attempt to 'solve' the system (4.47), (4.48) and (4.50) of BPS equations. Those equations are real linear in α and β. For this reason we can rescale α and β such that the normalization conditions (4.38) are nicer

$$A_1 = \alpha\alpha^* + \beta\beta^*, \qquad A_2 = \nu_1(\alpha\beta + \alpha^*\beta^*) \qquad \text{and} \qquad A_3 = i\nu_1\nu_2(\alpha\beta - \alpha^*\beta^*). \tag{5.1}$$

Those conditions will turn out to be crucial for "bootstrapping" the system.

In order to get a better understanding of what to expect from the general solution, let us first start by verifying that $AdS_5 \times S^5$ is a solution and where supersymmetric brane probes are sitting.

5.1 $AdS_5 \times S^5$

The pure $AdS_5 \times S^5$ solution is an $AdS_4 \times S^2 \times S^2$ fibration over an infinite strip. On the one boundary of the strip one S^2 is shrinking to zero size, whereas on the other boundary the other S^2 is shrinking to zero size. This can be seen by embedding AdS_5 into $\mathbb{R}^6$ spanned by $X_{-1}, X_0 \cdots, X_4$ and S^5 into $\mathbb{R}^6$ spanned by $Y_1, \ldots, Y_6$

$$-X_{-1}^2 - X_0^2 + X_1^2 + \cdots + X_4^2 = -R^2 \qquad \text{and} \qquad Y_1^2 + \cdots + Y_6^2 = R^2. \tag{5.2}$$

Then the strip can be parametrized by $-\infty < X_4 < \infty$ and $0 < r < R$ such that

$$Y_1^2 + Y_2^2 + Y_3^2 = r^2 \qquad \text{and} \qquad Y_4^2 + Y_5^2 + Y_6^2 = R^2 - r^2. \tag{5.3}$$

The solution has no 3-form flux, i.e. $g = h = 0$ and the dilatino variation equations (4.47) imply that the dilaton is constant $p = 0$. The gravitino variation equations (4.50) lead to the holomorphicity conditions

$$\partial_{\bar{z}}(\alpha^{*2} A_4) = \partial_{\bar{z}}(\beta^2 A_4) = \partial_{\bar{z}} \frac{\alpha\beta^*}{A_4} = 0. \tag{5.4}$$

This implies that $|\alpha|^2|\beta|^2$ is holomorphic and real, i.e.

$$|\alpha|^2|\beta|^2 = c^4. \tag{5.5}$$

Furthermore, $\alpha\beta|\beta|^2$ is holomorphic and (5.1) implies that it is real on one boundary of M_2 and imaginary on the other one. This determines[7]

$$\alpha\beta|\beta|^2 = c^4 e^z, \tag{5.6}$$

which implies

$$\alpha = c e^{-\frac{x}{2} + i\phi_\alpha} \qquad \text{and} \qquad \beta = c e^{\frac{x}{2} + iy - i\phi_\alpha}. \tag{5.7}$$

Using the equations (5.1) we can determine

$$A_1 = 2c^2 \cosh(x), \qquad A_2 = 2c^2 \nu_1 \cos(y) \qquad \text{and} \qquad A_3 = -2c^2 \nu_1 \nu_2 \sin(y). \tag{5.8}$$

For the range of $y \in [0, \frac{\pi}{2}]$ to make sense, we have to set $\nu_1 = 1$ and $\nu_2 = -1$. The gravitino variation equations (4.48) then imply

$$\alpha = c e^{-\frac{z^*}{2}}, \qquad \beta = c e^{\frac{z}{2}}, \qquad A_4 = 2c^2 \qquad \text{and} \qquad f = 60. \tag{5.9}$$

From this we conclude that

$$2c^2 = R. \tag{5.10}$$

The general solutions that we are looking for have to have this asymptotic form, i.e. they have to have semi-infinite strips where $A_2 = 0$ on one side and $A_3 = 0$ on the other side with a constant dilaton and no 3-form fluxes.

5.2 The general 'bootstrap'

Let us start by using (4.47)

$$\frac{g}{12} = p\left(\frac{\alpha}{\beta^*} - \frac{\beta^*}{\alpha}\right), \qquad \frac{ih}{12} = -p\left(\frac{\alpha}{\beta^*} + \frac{\beta^*}{\alpha}\right) \tag{5.11}$$

[7] One could choose a different holomorphic function, but the solution would still locally be $AdS_5 \times S^5$.

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and (5.1) to eliminate g, h, A_1, A_2 and A_3 from the BPS equations. The equations (4.48) allow to solve for A_4, f, p in terms of α and β

$$\begin{aligned}
\frac{\nu_2 A_4}{2\alpha\beta^*} &= \frac{4\frac{\alpha\beta}{\alpha^*\beta^*}-4\frac{\alpha^*\beta^*}{\alpha\beta}}{\left(\frac{\alpha}{\beta^*}\right)^2+\left(\frac{\beta^*}{\alpha}\right)^2-2\frac{\alpha\beta}{\alpha^*\beta^*}-2\frac{\alpha^*\beta^*}{\alpha\beta}}\,\partial_z \log|\alpha\beta| - \partial_z \log\left(\frac{\alpha\beta}{\alpha^*\beta^*}\right), \\
p &= -\frac{4}{\left(\frac{\alpha}{\beta^*}\right)^2+\left(\frac{\beta^*}{\alpha}\right)^2-2\frac{\alpha\beta}{\alpha^*\beta^*}-2\frac{\alpha^*\beta^*}{\alpha\beta}}\,\partial_z \log|\alpha\beta|, \\
\frac{f}{60} &= \partial_z \log\left(\frac{\alpha\beta}{\alpha^*\beta^*}\right) + 2\frac{\left(\frac{\alpha}{\beta^*}\right)^2-\left(\frac{\beta^*}{\alpha}\right)^2-2\frac{\alpha\beta}{\alpha^*\beta^*}+2\frac{\alpha^*\beta^*}{\alpha\beta}}{\left(\frac{\alpha}{\beta^*}\right)^2+\left(\frac{\beta^*}{\alpha}\right)^2-2\frac{\alpha\beta}{\alpha^*\beta^*}-2\frac{\alpha^*\beta^*}{\alpha\beta}}\,\partial_z \log|\alpha\beta|
\end{aligned} \tag{5.12}$$

and lead to one more independent equation for α and β

$$\frac{|\alpha|^2-|\beta|^2}{|\alpha|^2+|\beta|^2}\,\frac{\nu_2 A_4}{2\alpha\beta^*} - 2\partial_z \log(|\alpha|^2+|\beta|^2) + \frac{p}{2}\left(\left(\frac{\alpha}{\beta^*}\right)^2+\left(\frac{\beta^*}{\alpha}\right)^2\right) = 0. \tag{5.13}$$

The difference of the first two equations (4.50) leads to the identity

$$\frac{\left(\frac{\alpha}{\beta^*}\right)^2+\left(\frac{\beta^*}{\alpha}\right)^2-2\frac{\alpha\beta}{\alpha^*\beta^*}-2\frac{\alpha^*\beta^*}{\alpha\beta}}{\left(\frac{\alpha}{\beta^*}\right)^2-\left(\frac{\beta^*}{\alpha}\right)^2}\,\partial_z \log\left(\frac{\alpha}{\beta^*}\right) = -2\partial_z \log|\alpha\beta| \tag{5.14}$$

and the sum of the first two equations (4.50) leads to

$$\partial_z\left(\frac{\alpha^2\beta^{*2}A_4^2}{\left(\frac{\alpha}{\beta^*}\right)^2-\left(\frac{\beta^*}{\alpha}\right)^2}\right) = 0, \tag{5.15}$$

which implies that

$$\frac{\alpha^2\beta^{*2}A_4^2}{\left(\frac{\alpha}{\beta^*}\right)^2-\left(\frac{\beta^*}{\alpha}\right)^2} = a(z^*), \tag{5.16}$$

where $a(z^*)$ is an antiholomorphic function. The other two equations (4.50) are redundant.

We can reexpress $\frac{\nu_2 A_4}{2\alpha\beta^*}$ and p in terms of the ratio $\frac{\alpha}{\beta^*}$

$$\begin{aligned}
\frac{\nu_2 A_4}{2\alpha\beta^*} &= -2\frac{\frac{\alpha\beta}{\alpha^*\beta^*}-\frac{\alpha^*\beta^*}{\alpha\beta}}{\left(\frac{\alpha}{\beta^*}\right)^2-\left(\frac{\beta^*}{\alpha}\right)^2}\,\partial_z \log\left(\frac{\alpha}{\beta^*}\right) - \partial_z \log\left(\frac{\alpha\beta}{\beta^*\alpha^*}\right), \\
p &= \frac{2}{\left(\frac{\alpha}{\beta^*}\right)^2-\left(\frac{\beta^*}{\alpha}\right)^2}\,\partial_z \log\left(\frac{\alpha}{\beta^*}\right).
\end{aligned} \tag{5.17}$$

The equations (5.13), (5.14) and (5.16) are then the remaining system of equations for α and β. Actually, (5.13) only depends on the ratio $\frac{\alpha}{\beta^*}$ and turns out to be the last equation that is trivially satisfied. The other two equations form a second order system for α and β.

Those equations are algebraic in the phase of $\alpha\beta^*$. Equation (5.16) can be solved for $|\alpha\beta^*|$ in terms of $\frac{\alpha}{\beta^*}$, this can be inserted into (5.14) to give a single second order differential equation for $\frac{\alpha}{\beta^*}$.

5.3 Probe branes and boundary conditions

To start understanding the general solution let us first look at the probe branes again. The projector equation for a supersymmetric NS5-brane around S^2 with k units of magnetic flux is [29]

$$\frac{i}{\sqrt{1+\left(\frac{\pi k}{R}\right)^2}}\gamma^{(1)}\gamma^{(2)} * \epsilon - \frac{\frac{\pi k}{R}}{\sqrt{1+\left(\frac{\pi k}{R}\right)^2}}\gamma^{(1)}\epsilon = \epsilon, \tag{5.18}$$

and the projector equation for a supersymmetric D5-brane around $\tilde{S}^2$ with k units of magnetic flux is

$$-\frac{1}{\sqrt{1+\left(\frac{\pi k}{R}\right)^2}}\gamma^{(1)}\gamma^{(3)} * \epsilon - \frac{\frac{\pi k}{R}}{\sqrt{1+\left(\frac{\pi k}{R}\right)^2}}\gamma^{(1)}\epsilon = \epsilon. \tag{5.19}$$

Those equations turn into

$$e^{z^*_{NS5_k}} = \frac{\beta^*}{\alpha} = \sqrt{1+\left(\frac{\pi k}{R}\right)^2} + \frac{\pi k}{R} \quad \text{and} \quad e^{z^*_{D5_k}} = \frac{\beta^*}{\alpha} = -i\left(\sqrt{1+\left(\frac{\pi k}{R}\right)^2} + \frac{\pi k}{R}\right) \tag{5.20}$$

which is

$$\sinh(x(k)) = \frac{\pi k}{R}, \qquad y_{NS5} = 0 \qquad \text{or} \qquad y_{D5} = \frac{\pi}{2} \tag{5.21}$$

in agreement with the predictions of section 3. From this it is easy to see that the NS5-branes are sitting in a place where $\frac{\alpha}{\beta^*}$ is real, whereas the D5-branes are sitting in a place where $\frac{\alpha}{\beta^*}$ is imaginary. The absolute value $\left|\frac{\alpha}{\beta^*}\right|$ determines the magnetic flux on the brane.

For the $AdS_5 \times S^5$ solution this means that the NS5-branes are sitting on the boundary of the strip where S^2 has maximal size, call it the 'black' boundary and the D5-branes are sitting on the boundary of the strip where $\tilde{S}^2$ has maximal size, call it the 'white' boundary. In the regions where the 5-branes are sitting, we expect a backreaction of the geometry, which generates the throat of a 5-brane. This means that there is a 3-sphere which supports the appropriate 3-form flux. This is done by switching the shrunk 2-sphere in the respective region of the boundary. To understand this better, we need to work out the boundary conditions in the different regions.

The two dimensional geometry can be conformally mapped to a region in the complex plane. This region has a boundary on which one of the two 2-spheres is shrinking to zero size. In order to parametrize the boundary in a more invariant way, we impose the boundary condition

$$A_4|_{\partial M_2} = 1. \tag{5.22}$$

The boundary is divided into (colored) segments on which either one or the other 2-sphere is shrinking to zero size

$$A_2|_{\partial M_2,w} = 0 \qquad \text{or} \qquad A_3|_{\partial M_2,b} = 0. \tag{5.23}$$

Using (5.1) this leads to the same conditions on the phase of $\frac{\alpha}{\beta^*}$ as the five-brane projectors do. Furthermore, in order for the geometry to be smooth, one has to require either

$$\partial_n A_1|_{\partial M_2,w} = 0, \qquad \partial_n A_2|_{\partial M_2,w} = 1, \qquad \partial_n A_3|_{\partial M_2,w} = 0, \qquad \text{and} \qquad \partial_n A_4|_{\partial M_2,w} = 0, \tag{5.24}$$

or alternatively

$$\partial_n A_1|_{\partial M_2,b} = 0, \qquad \partial_n A_2|_{\partial M_2,b} = 0, \qquad \partial_n A_3|_{\partial M_2,b} = 1, \qquad \text{and} \qquad \partial_n A_4|_{\partial M_2,b} = 0, \tag{5.25}$$

where ∂_n is the normal derivative to the boundary. In order for the fluxes to be regular, we either need to require

$$p_n|_{\partial M_2,w} = f_n|_{\partial M_2,w} = g_t|_{\partial M_2,w} = h_n|_{\partial M_2,w} = 0 \tag{5.26}$$

or

$$p_n|_{\partial M_2,b} = f_n|_{\partial M_2,b} = g_n|_{\partial M_2,b} = h_t|_{\partial M_2,b} = 0. \tag{5.27}$$

It is not difficult to see that the boundary conditions on g and h are satisfied, once the boundary condition on p is satisfied.

Let us concentrate on the 'white' boundary, where $A_2|_{\partial M_2,w} = 0$. We assume that it is along the x-axis and that the strip is on the upper half plane. There the boundary conditions imply for the spinor variables $\frac{\alpha}{\beta^*}$ and $\alpha\beta^*$

$$\begin{aligned} &\tfrac{\alpha}{\beta^*}|_{\partial M_2,w} \in i\mathbb{R}, \qquad \alpha\beta^*|_{\partial M_2,w} \in \mathbb{R}, \qquad \left(\tfrac{\alpha}{\beta^*}\left|\tfrac{\beta^*}{\alpha}\right|\tfrac{|\alpha\beta^*|}{\alpha\beta^*}\right)_{\partial M_2,w} = i\nu_1\nu_2, \\ &\partial_y\left(\tfrac{\alpha}{\beta^*}\right)_{\partial M_2,w} = \tfrac{\nu_1}{2}\left(\left|\tfrac{\alpha}{\beta^*}\right|\tfrac{1}{|\alpha\beta^*|}\right)_{\partial M_2,w}, \quad \partial_y|\alpha\beta^*|_{\partial M_2,w} = 0. \end{aligned} \tag{5.28}$$

Note that all the normal derivatives of the spinor variables are determined, except for $\partial_y \arg(\alpha\beta^*)|_{\partial M_2,w}$. This phase only appears in the expression for A_4 in terms of the spinor variables.

The antiholomorphic function $a(z^*)$ has to be real on this boundary. Given $a(z^*)$ and using both boundary conditions on A_4, (5.16) can be solved for $|\alpha\beta^*|$

$$|\alpha\beta^*|_{\partial M_2,w} = |a|\left|\left(\frac{\alpha}{\beta^*}\right)^2 - \left(\frac{\beta^*}{\alpha}\right)^2\right|. \tag{5.29}$$

This can be inserted into (5.14) to give a second order ODE for $\frac{\alpha}{\beta^*}$. The solutions of that ODE are determined by the values of $\frac{\alpha}{\beta^*}$ at the 'ends' of the 'white' boundary. The above boundary conditions are then enough for the second order PDE of 5.2. The antiholomorphic function presumably has to be determined by the reality condition above and its asymptotic behavior.

Similarly the boundary conditions on the spinor variables for a 'black' boundary along the y-axis, where the strip is the right half plane are

$$\begin{aligned} &\tfrac{\alpha}{\beta^*}|_{\partial M_2,b} \in \mathbb{R}, \qquad \alpha\beta^*|_{\partial M_2,b} \in i\mathbb{R}, \qquad \left(\tfrac{\alpha}{\beta^*}\left|\tfrac{\beta^*}{\alpha}\right|\tfrac{|\alpha\beta^*|}{\alpha\beta^*}\right)_{\partial M_2,b} = i\nu_1, \\ &\partial_x\left(\tfrac{\alpha}{\beta^*}\right)_{\partial M_2,b} = -\tfrac{i\nu_1\nu_2}{2}\left(\left|\tfrac{\alpha}{\beta^*}\right|\tfrac{1}{|\alpha\beta^*|}\right)_{\partial M_2,b}, \quad \partial_x|\alpha\beta^*|_{\partial M_2,b} = 0. \end{aligned} \tag{5.30}$$

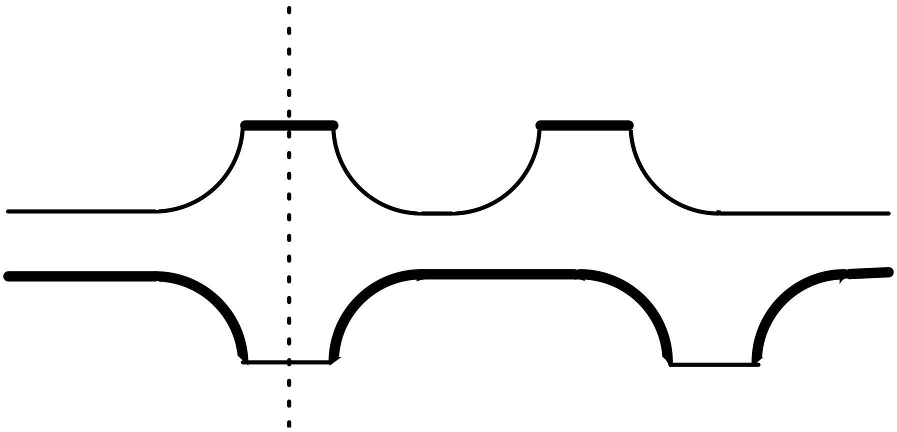

Figure 3: A possibility for the backreacted geometry.

Let us try to see how this story fits in with the probe brane picture. We expect that the five-branes get replaced by geometry with fluxes. At the position of the defect we expect that the value of $\frac{\alpha}{\beta^*}$ agrees with the one from the probe brane calculation. There are two possibilities that can happen: The defect is either a finite or an infinite distance along the boundary away from a given reference point. Furthermore the geometry at the defect has to have a 3-cycle that supports the flux.

For a defect at finite distance this can be done by a change of coloring, i.e. by inserting a finite interval of 'white' boundary into the 'black' boundary or vice versa. This creates a 3-sphere which can support the flux. On the 'black' side of the interface, the value of $\frac{\alpha}{\beta^*}$ is given by the probe brane value.

In order for the geometry to be smooth at the interface of a 'black' and a 'white' boundary, it needs to have a right angle, such that the strip turns locally into a quadrant. Unlike the cases of chiral operators or Wilson lines [15, 16, 20], there are no such interfaces in our vacuum ($AdS_5 \times S^5$) solution. Actually, closer examination reveals that not all the regularity conditions can hold at the same time. For example, if the boundary conditions on A_2 and A_3 hold at the same time, then $\partial_z \log |\alpha\beta|$ diverges at the interface. This implies that the dilaton p diverges or $\frac{\alpha}{\beta^*}$ diverges.

On the other hand a defect at infinite distance produces an infinite throat with the same color on both sides. This also creates a three-sphere to support the three-form flux. We expect that $\frac{\alpha}{\beta^*}$ asymptotes to the value given by the probe brane picture.

In the asymptotic region of such a throat we expect $\frac{\alpha}{\beta^*}$ to be almost constant at the boundary. This implies that $\alpha\beta^*$ is linearly growing at the boundary, i.e. the warp factors A_1 and A_3 are growing linearly and the dilaton is growing logarithmically along the boundary. In the other direction this is of course bounded and cannot continue forever, it has to connect to an asymptotically $AdS_5 \times S^5$ region.

As of now we haven't found any convincing argument for either scenario, but those seem to be the only possibilities for the backreacted geometry of a five-brane.

We believe that those difficulties are arising due to the fact that we are trying to describe the backreacted geometry of five-branes instead of D3-branes [15] or strings [16, 20], where the geometry seems to be really well behaved at the locations of the 'defects'.

The gravity discussion also leaves open the possiblity of more than two asymptotic $AdS_5 \times S^5$ regions. This would correspond to several $\mathcal{N} = 4$ super Yang Mills theories that interact on a defect. In the case of only a single asymptotic $AdS_5 \times S^5$ region one would get a $\mathcal{N} = 4$ super Yang Mills theory with a boundary. The latter case requires an interface between a 'black' and a 'white' boundary.

There is more work to be done in order to understand those outstanding issues better. To complete the story, one also needs to calculate the fluxes through all the three-cycles as well as change of the rank of the gauge group. Those impose the true physical boundary conditions and might be calculable even if the full solution is not known (see e.g. [28]). We leave a closer examination of all those issues for future work [30].

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A. Clifford algebra conventions

A.1 Generalities

The Clifford algebra is defined by the anticommutation relations

$$\{\gamma^m, \gamma^n\} = 2\eta^{mn}, \tag{A.1}$$

where $\eta^{mn} = \eta^m \delta^{mn}$. We choose a representation in which $\sqrt{\eta^m}\gamma^m$ is Hermitean[8]. Given a complex structure, one can define the raising and lowering operators

$$\Gamma^m = \sqrt{\eta^{2m}}\gamma^{2m} + i\sqrt{\eta^{2m+1}}\gamma^{2m+1}, \quad \text{and} \quad (\Gamma^m)^\dagger = \sqrt{\eta^{2m}}\gamma^{2m} - i\sqrt{\eta^{2m+1}}\gamma^{2m+1}. \tag{A.2}$$

Then the raising and lowering operators satisfy the following anticommutation relations:

$$\{\Gamma^m, \Gamma^n\} = \{(\Gamma^m)^\dagger, (\Gamma^n)^\dagger\} = 0 \quad \text{and} \quad \{\Gamma^m, (\Gamma^n)^\dagger\} = 4\delta^{mn}. \tag{A.3}$$

One can then define the fermion number operators

$$F^m = i\sqrt{\eta^{2m}}\gamma^{2m}\sqrt{\eta^{2m+1}}\gamma^{2m+1} = 1 - \frac{1}{2}\Gamma^m(\Gamma^m)^\dagger = -1 + \frac{1}{2}(\Gamma^m)^\dagger\Gamma^m. \tag{A.4}$$

The chirality operator is then the product of all the Fermion number operators $\gamma = F^1 \cdots F^n$.

[8] By the square root we mean $\sqrt{1} = 1$ and $\sqrt{-1} = i$.

The Fermion number operators have eigenvalues ± 1. The eigenvalues of the Fermion number operators can be used to label a basis of states. One can define a ground state $|0\rangle$ which is anihilated by all the lowering operators. It has Fermion number -1 for all Fermion number operators. All other states can be gotten by applying raising operators. If one labels a state by $|\nu_1, \ldots, \nu_n\rangle$, then the raising and lowering operators act as follows:

$$|\nu_1, \ldots, +1, \ldots, \nu_n\rangle = \frac{1}{2}\nu_1 \cdots \nu_{m-1}(\Gamma^m)^\dagger |\nu_1, \ldots, -1, \ldots, \nu_n\rangle, \tag{A.5}$$

$$|\nu_1, \ldots, -1, \ldots, \nu_n\rangle = \frac{1}{2}\nu_1 \cdots \nu_{m-1}\Gamma^m |\nu_1, \ldots, +1, \ldots, \nu_n\rangle. \tag{A.6}$$

This defines the matrix elements of the gamma matrices. One can see that in this basis Γ^m is real. From this follows that

- the matrices $\sqrt{\eta^m}\gamma^m$ are Hermitean,
- the matrices $\sqrt{\eta^{2m}}\gamma^{2m}$ are symmetric and real and
- the matrices $\sqrt{\eta^{2m+1}}\gamma^{2m+1}$ are antisymmetric and imaginary.

In general there are matrices B, C and D such that

$$(\gamma^m)^* = \eta_B B \gamma^m B^{-1}, \tag{A.7}$$

$$(\gamma^m)^\dagger = \eta_C C \gamma^m C^{-1}, \tag{A.8}$$

$$(\gamma^m)^t = \eta_D D \gamma^m D^{-1}, \tag{A.9}$$

where $\eta_B, \eta_C, \eta_D = \pm 1$ is a constant. Given a spinor ϵ, $*\epsilon = B^{-1}\epsilon^*$, $\bar{\epsilon} = \epsilon^\dagger C$ and $\tilde{\epsilon} = \epsilon^t D$ transform covariantly.

If $BB^* = \mathbb{1}$ one can impose the Majorana condition $\epsilon = *\epsilon$. And if B commutes with the chirality operator γ, one can impose the Majorana-Weyl condition.

A.2 Spin(1, 9)

Chirality operator:

$$\gamma^{(10)} = \gamma^{0\cdots 9} \tag{A.10}$$

Complex conjugation:

$$B^{(10)} = \gamma^{013579} \tag{A.11}$$

$$B^{(10)}\gamma^M (B^{(10)})^{-1} = (\gamma^M)^* \tag{A.12}$$

Hermitean conjugation:

$$C^{(10)} = \gamma^0 \tag{A.13}$$

$$C^{(10)}\gamma^M (C^{(10)})^{-1} = -(\gamma^M)^\dagger \tag{A.14}$$

Transpose:

$$D^{(10)} = \gamma^{13579} \tag{A.15}$$

$$D^{(10)}\gamma^M (D^{(10)})^{-1} = -(\gamma^M)^t \tag{A.16}$$

A.3 Spin(1,3) — AdS_4

$$\gamma^{(1)} = i\gamma^{0123} \tag{A.17}$$

Complex Conjugation:

$$B^{(1)} = \gamma^2 \tag{A.18}$$

$$B^{(1)}\gamma^\mu(B^{(1)})^{-1} = (\gamma^\mu)^* \qquad \text{and} \qquad B^{(1)}(i\gamma^{(1)})(B^{(1)})^{-1} = (i\gamma^{(1)})^* \tag{A.19}$$

$$B^{(1)}(B^{(1)})^* = \mathbb{1} \tag{A.20}$$

Hermitean conjugation:

$$C^{(1)} = \gamma^{123} \tag{A.21}$$

$$C^{(1)}\gamma^\mu(C^{(1)})^{-1} = (\gamma^\mu)^\dagger \qquad \text{and} \qquad C^{(1)}(i\gamma^{(1)})(C^{(1)})^{-1} = (i\gamma^{(1)})^\dagger \tag{A.22}$$

Transpose:

$$D^{(1)} = \gamma^{13} \tag{A.23}$$

$$D^{(1)}\gamma^\mu(D^{(1)})^{-1} = (\gamma^\mu)^t \qquad \text{and} \qquad D^{(1)}(i\gamma^{(1)})(D^{(1)})^{-1} = (i\gamma^{(1)})^t \tag{A.24}$$

A.4 Spin(2) — S^2

$$\gamma^{(2)} = i\gamma^{45} \tag{A.25}$$

Complex Conjugation:

$$B^{(2)} = \gamma^5 \tag{A.26}$$

$$B^{(2)}\gamma^m(B^{(2)})^{-1} = -(\gamma^m)^* \qquad \text{and} \qquad B^{(2)}\gamma^{(2)}(B^{(2)})^{-1} = -(\gamma^{(2)})^* \tag{A.27}$$

$$B^{(2)}(B^{(2)})^* = -\mathbb{1} \tag{A.28}$$

Hermitean conjugation:

$$C^{(2)} = \mathbb{1} \tag{A.29}$$

$$C^{(2)}\gamma^m(C^{(2)})^{-1} = (\gamma^m)^\dagger \qquad \text{and} \qquad C^{(2)}\gamma^{(2)}(C^{(2)})^{-1} = (\gamma^{(2)})^\dagger \tag{A.30}$$

Transpose:

$$D^{(2)} = \gamma^5 \tag{A.31}$$

$$D^{(2)}\gamma^m(D^{(2)})^{-1} = -(\gamma^m)^t \qquad \text{and} \qquad D^{(2)}\gamma^{(2)}(D^{(2)})^{-1} = -(\gamma^{(2)})^t \tag{A.32}$$

Similarly

$$\gamma^{(3)} = i\gamma^{67}, \qquad B^{(3)} = \gamma^7, \qquad C^{(3)} = \mathbb{1} \qquad \text{and} \qquad D^{(3)} = \gamma^7. \tag{A.33}$$

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A.5 Spin(2) — M_2

$$\gamma^{(2)} = i\gamma^{89} \tag{A.34}$$

Complex Conjugation:

$$B^{(4)} = \gamma^8 \tag{A.35}$$

$$B^{(4)}\gamma^a(B^{(4)})^{-1} = (\gamma^a)^* \tag{A.36}$$

$$B^{(4)}(B^{(4)})^* = \mathbb{1} \tag{A.37}$$

Hermitean conjugation:

$$C^{(4)} = \mathbb{1} \tag{A.38}$$

$$C^{(4)}\gamma^a(C^{(4)})^{-1} = (\gamma^m)^\dagger \tag{A.39}$$

Transpose:

$$D^{(4)} = \gamma^8 \tag{A.40}$$

$$D^{(4)}\gamma^a(D^{(4)})^{-1} = (\gamma^a)^t \tag{A.41}$$

A.6 Decomposition of a 10-dimensional spinor

The ten dimensional gamma matrix algebra can be decomposed in the folowing way

$$\begin{aligned}
&\gamma^\mu = \gamma^\mu \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1},\\
&\gamma^m = \gamma^{(1)} \otimes \gamma^m \otimes \mathbb{1} \otimes \mathbb{1},\\
&\gamma^i = \gamma^{(1)} \otimes \gamma^{(2)} \otimes \gamma^i \otimes \mathbb{1},\\
&\gamma^a = \gamma^{(1)} \otimes \gamma^{(2)} \otimes \gamma^{(3)} \otimes \gamma^a,\\
&\gamma^{(10)} = \gamma^{(1)} \otimes \gamma^{(2)} \otimes \gamma^{(3)} \otimes \gamma^{(4)},\\
&B^{(10)} = -B^{(1)} \otimes B^{(2)} \otimes (B^{(3)}\gamma^{(3)}) \otimes (B^{(4)}\gamma^{(4)}),\\
&C^{(10)} = -i(C^{(1)}\gamma^{(1)}) \otimes C^{(2)} \otimes C^{(3)} \otimes C^{(4)},\\
&D^{(10)} = -i(D^{(1)}\gamma^{(1)}) \otimes D^{(2)} \otimes (D^{(3)}\gamma^{(3)}) \otimes (D^{(4)}\gamma^{(4)}).
\end{aligned} \tag{A.42}$$

B. Relations for spinor bilinears

In this appendix we summarize some properties of spinor bilinears.

$$(\bar\epsilon_2\gamma^M\epsilon_1)^\dagger = \bar\epsilon_1\gamma^M\epsilon_2, \tag{B.1}$$

$$\tilde\epsilon_1\gamma^M\epsilon_2 = -\tilde\epsilon_2\gamma^M\epsilon_1, \tag{B.2}$$

$$\overline{*\epsilon_1}\gamma^M * \epsilon_2 = \bar\epsilon_2\gamma^M\epsilon_1, \tag{B.3}$$

$$\overline{*\epsilon_1}\gamma^M\epsilon_2 = \tilde\epsilon_1\gamma^M\epsilon_2. \tag{B.4}$$

The following table summarizes the symmetry properties of 8-dimensional spinor bilinears under transpositio(exchange of $(\alpha\dot\alpha)$ and $(\beta\dot\beta)$) and complex conjugation

bilinear	t	$*$
$\bar{\chi}^{(1,1,1)}_{\alpha\dot{\alpha}}((\gamma^{(1)\frac{\eta_1+\eta_1'}{2}}\gamma^{\mu})\otimes(\gamma^{(2)\frac{\eta_2-\eta_2'}{2}})\otimes(\gamma^{(3)\frac{\eta_3-\eta_3'}{2}}))\chi^{(1,1,1)}_{\beta\dot{\beta}}$	$\eta_1\eta_1'\eta_2\eta_2'\eta_3\eta_3'$	$-\eta\eta'$
$\bar{\chi}^{(1,1,1)}_{\alpha\dot{\alpha}}((\gamma^{(1)\frac{\eta_1-\eta_1'}{2}})\otimes(\gamma^{(2)\frac{\eta_2-\eta_2'}{2}}\gamma^{m})\otimes(\gamma^{(3)\frac{\eta_3-\eta_3'}{2}}))\chi^{(1,1,1)}_{\beta\dot{\beta}}$	$\eta_3\eta_3'$	$-\eta\eta'$
$\bar{\chi}^{(1,1,1)}_{\alpha\dot{\alpha}}((\gamma^{(1)\frac{\eta_1-\eta_1'}{2}})\otimes(\gamma^{(2)\frac{\eta_2+\eta_2'}{2}})\otimes(\gamma^{(3)\frac{\eta_3-\eta_3'}{2}}\gamma^{i}))\chi^{(1,1,1)}_{\beta\dot{\beta}}$	$-\eta_2\eta_2'$	$\eta\eta'$
$\bar{\chi}^{(1,1,1)}_{\alpha\dot{\alpha}}((\gamma^{(1)\frac{\eta_1-\eta_1'}{2}})\otimes(\gamma^{(2)\frac{\eta_2+\eta_2'}{2}})\otimes(\gamma^{(3)\frac{\eta_3+\eta_3'}{2}}))\chi^{(1,1,1)}_{\beta\dot{\beta}}$	$-\eta_2\eta_2'\eta_3\eta_3'$	$\eta\eta'$
$\bar{\chi}^{(1,1,1)}_{\alpha\dot{\alpha}}((\gamma^{(1)\frac{\eta_1+\eta_1'}{2}}\gamma^{\mu\nu\rho})\otimes(\gamma^{(2)\frac{\eta_2-\eta_2'}{2}})\otimes(\gamma^{(3)\frac{\eta_3-\eta_3'}{2}}))\chi^{(1,1,1)}_{\beta\dot{\beta}}$	$-\eta_1\eta_1'\eta_2\eta_2'\eta_3\eta_3'$	$-\eta\eta'$
$\bar{\chi}^{(1,1,1)}_{\alpha\dot{\alpha}}((\gamma^{(1)\frac{\eta_1-\eta_1'}{2}}\gamma^{\mu\nu})\otimes(\gamma^{(2)\frac{\eta_2+\eta_2'}{2}})\otimes(\gamma^{(3)\frac{\eta_3+\eta_3'}{2}}))\chi^{(1,1,1)}_{\beta\dot{\beta}}$	$\eta_2\eta_2'\eta_3\eta_3'$	$\eta\eta'$
$\bar{\chi}^{(1,1,1)}_{\alpha\dot{\alpha}}((\gamma^{(1)\frac{\eta_1-\eta_1'}{2}})\otimes(\gamma^{(2)\frac{\eta_2+\eta_2'}{2}}\gamma^{mn})\otimes(\gamma^{(3)\frac{\eta_3+\eta_3'}{2}}))\chi^{(1,1,1)}_{\beta\dot{\beta}}$	$\eta_2\eta_2'\eta_3\eta_3'$	$\eta\eta'$
$\bar{\chi}^{(1,1,1)}_{\alpha\dot{\alpha}}((\gamma^{(1)\frac{\eta_1-\eta_1'}{2}})\otimes(\gamma^{(2)\frac{\eta_2+\eta_2'}{2}})\otimes(\gamma^{(3)\frac{\eta_3+\eta_3'}{2}}\gamma^{ij}))\chi^{(1,1,1)}_{\beta\dot{\beta}}$	$\eta_2\eta_2'\eta_3\eta_3'$	$\eta\eta'$
$\bar{\chi}^{(1,1,1)}_{\alpha\dot{\alpha}}((\gamma^{(1)\frac{\eta_1-\eta_1'}{2}})\otimes(\gamma^{(2)\frac{\eta_2-\eta_2'}{2}}\gamma^{m})\otimes(\gamma^{(3)\frac{\eta_3+\eta_3'}{2}}\gamma^{i}))\chi^{(1,1,1)}_{\beta\dot{\beta}}$	-1	$-\eta\eta'$
$\bar{\chi}^{(1,1,1)}_{\alpha\dot{\alpha}}((\gamma^{(1)\frac{\eta_1-\eta_1'}{2}})\otimes(\gamma^{(2)\frac{\eta_2+\eta_2'}{2}}\gamma^{mn})\otimes(\gamma^{(3)\frac{\eta_3-\eta_3'}{2}}\gamma^{i}))\chi^{(1,1,1)}_{\beta\dot{\beta}}$	$\eta_2\eta_2'$	$\eta\eta'$
$\bar{\chi}^{(1,1,1)}_{\alpha\dot{\alpha}}((\gamma^{(1)\frac{\eta_1-\eta_1'}{2}})\otimes(\gamma^{(2)\frac{\eta_2-\eta_2'}{2}}\gamma^{m})\otimes(\gamma^{(3)\frac{\eta_3-\eta_3'}{2}}\gamma^{ij}))\chi^{(1,1,1)}_{\beta\dot{\beta}}$	$-\eta_3\eta_3'$	$-\eta\eta'$
$\bar{\chi}^{(1,1,1)}_{\alpha\dot{\alpha}}((\gamma^{(1)\frac{\eta_1-\eta_1'}{2}}\gamma^{\mu\nu})\otimes(\gamma^{(2)\frac{\eta_2+\eta_2'}{2}}\gamma^{mn})\otimes(\gamma^{(3)\frac{\eta_3-\eta_3'}{2}}\gamma^{i}))\chi^{(1,1,1)}_{\beta\dot{\beta}}$	$-\eta_2\eta_2'$	$\eta\eta'$
$\bar{\chi}^{(1,1,1)}_{\alpha\dot{\alpha}}((\gamma^{(1)\frac{\eta_1-\eta_1'}{2}}\gamma^{\mu\nu})\otimes(\gamma^{(2)\frac{\eta_2-\eta_2'}{2}}\gamma^{m})\otimes(\gamma^{(3)\frac{\eta_3-\eta_3'}{2}}\gamma^{ij}))\chi^{(1,1,1)}_{\beta\dot{\beta}}$	$\eta_3\eta_3'$	$-\eta\eta'$
$\bar{\chi}^{(1,1,1)}_{\alpha\dot{\alpha}}((\gamma^{(1)\frac{\eta_1+\eta_1'}{2}}\gamma^{\mu})\otimes(\gamma^{(2)\frac{\eta_2-\eta_2'}{2}}\gamma^{mn})\otimes(\gamma^{(3)\frac{\eta_3-\eta_3'}{2}}\gamma^{ij}))\chi^{(1,1,1)}_{\beta\dot{\beta}}$	$\eta_1\eta_1'\eta_2\eta_2'\eta_3\eta_3'$	$-\eta\eta'$
$\bar{\chi}^{(1,1,1)}_{\alpha\dot{\alpha}}((\gamma^{(1)\frac{\eta_1-\eta_1'}{2}}\gamma^{\mu\nu})\otimes(\gamma^{(2)\frac{\eta_2+\eta_2'}{2}}\gamma^{mn})\otimes(\gamma^{(3)\frac{\eta_3+\eta_3'}{2}}))\chi^{(1,1,1)}_{\beta\dot{\beta}}$	$-\eta_2\eta_2'\eta_3\eta_3'$	$\eta\eta'$
$\bar{\chi}^{(1,1,1)}_{\alpha\dot{\alpha}}((\gamma^{(1)\frac{\eta_1-\eta_1'}{2}}\gamma^{\mu\nu})\otimes(\gamma^{(2)\frac{\eta_2+\eta_2'}{2}})\otimes(\gamma^{(3)\frac{\eta_3+\eta_3'}{2}}\gamma^{ij}))\chi^{(1,1,1)}_{\beta\dot{\beta}}$	$-\eta_2\eta_2'\eta_3\eta_3'$	$\eta\eta'$
$\bar{\chi}^{(1,1,1)}_{\alpha\dot{\alpha}}((\gamma^{(1)\frac{\eta_1+\eta_1'}{2}}\gamma^{\mu})\otimes(\gamma^{(2)\frac{\eta_2-\eta_2'}{2}}\gamma^{mn})\otimes(\gamma^{(3)\frac{\eta_3+\eta_3'}{2}}\gamma^{i}))\chi^{(1,1,1)}_{\beta\dot{\beta}}$	$\eta_1\eta_1'\eta_2\eta_2'$	$-\eta\eta'$
$\bar{\chi}^{(1,1,1)}_{\alpha\dot{\alpha}}((\gamma^{(1)\frac{\eta_1+\eta_1'}{2}}\gamma^{\mu})\otimes(\gamma^{(2)\frac{\eta_2+\eta_2'}{2}}\gamma^{m})\otimes(\gamma^{(3)\frac{\eta_3+\eta_3'}{2}}\gamma^{ij}))\chi^{(1,1,1)}_{\beta\dot{\beta}}$	$-\eta_1\eta_1'\eta_3\eta_3'$	$\eta\eta'$
$\bar{\chi}^{(1,1,1)}_{\alpha\dot{\alpha}}((\gamma^{(1)\frac{\eta_1+\eta_1'}{2}}\gamma^{\mu})\otimes(\gamma^{(2)\frac{\eta_2-\eta_2'}{2}}\gamma^{mn})\otimes(\gamma^{(3)\frac{\eta_3-\eta_3'}{2}}))\chi^{(1,1,1)}_{\beta\dot{\beta}}$	$-\eta_1\eta_1'\eta_2\eta_2'\eta_3\eta_3'$	$-\eta\eta'$
$\bar{\chi}^{(1,1,1)}_{\alpha\dot{\alpha}}((\gamma^{(1)\frac{\eta_1+\eta_1'}{2}}\gamma^{\mu})\otimes(\gamma^{(2)\frac{\eta_2-\eta_2'}{2}})\otimes(\gamma^{(3)\frac{\eta_3-\eta_3'}{2}}\gamma^{ij}))\chi^{(1,1,1)}_{\beta\dot{\beta}}$	$-\eta_1\eta_1'\eta_2\eta_2'\eta_3\eta_3'$	$-\eta\eta'$
$\bar{\chi}^{(1,1,1)}_{\alpha\dot{\alpha}}((\gamma^{(1)\frac{\eta_1-\eta_1'}{2}})\otimes(\gamma^{(2)\frac{\eta_2+\eta_2'}{2}}\gamma^{mn})\otimes(\gamma^{(3)\frac{\eta_3-\eta_3'}{2}}\gamma^{i}))\chi^{(1,1,1)}_{\beta\dot{\beta}}$	$\eta_2\eta_2'$	$\eta\eta'$
$\bar{\chi}^{(1,1,1)}_{\alpha\dot{\alpha}}((\gamma^{(1)\frac{\eta_1-\eta_1'}{2}})\otimes(\gamma^{(2)\frac{\eta_2-\eta_2'}{2}}\gamma^{m})\otimes(\gamma^{(3)\frac{\eta_3-\eta_3'}{2}}\gamma^{ij}))\chi^{(1,1,1)}_{\beta\dot{\beta}}$	$-\eta_3\eta_3'$	$-\eta\eta'$

(B.5)

where $\eta\eta' = \eta_1\eta_1'\eta_2\eta_2'\eta_3\eta_3'$.

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